Morphisms of projective varieties

Consider the function $\mathbb{P}^1 \to \mathbb{P}^2$ defined

$$[s:t] \mapsto [s^2 : st : t^2]$$

For $[s:t] \in \mathbb{P}^1$ and $\lambda \neq 0$,

$[s:t] = [\lambda s : \lambda t] \longmapsto [\lambda^2 s^2 : \lambda^2 s t : \lambda^2 t^2] = [s^2 : st : t^2]$, so this map is well-defined.

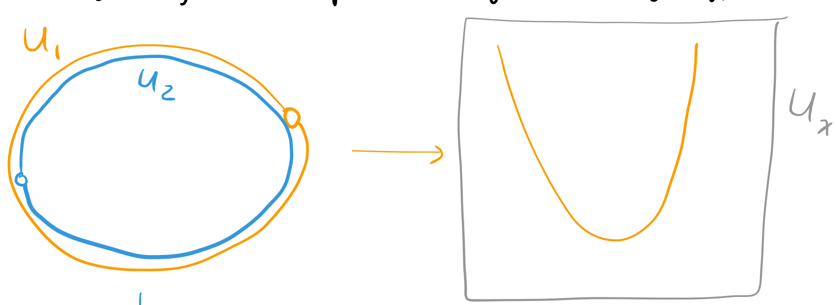
Since $s^2 t^2 = (st)^2$, the image of this map lies in $\overset{C}{\overset{\shortparallel}{V(xz - y^2)}} \subseteq \mathbb{P}^1$, so this defines a map $\mathbb{P}^1 \to C$.

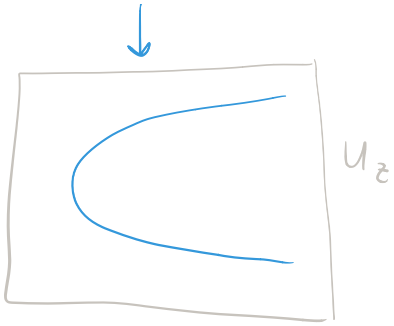
In $U_1 \subseteq \mathbb{P}^1$, the map becomes $[1 : \frac{t}{s}] \longmapsto [1 : (\frac{t}{s}) : (\frac{t}{s})^2] \subseteq \underset{\{[1:y:z]\}}{\underset{\shortparallel}{U_x}} \subseteq \mathbb{P}^2$

i.e. it corresponds to the map $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$

$$\alpha \longmapsto (\alpha : \alpha^2)$$

In $U_2 \subseteq \mathbb{P}^1$, the map is $[\frac{s}{t} : 1] \longmapsto [(\frac{s}{t})^2 : (\frac{s}{t}) : 1] \subseteq U_z \subseteq \mathbb{P}^2$.





i.e. this map restricts locally to a morphism of affine varieties.

Def: Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be projective algebraic sets. A function $\varphi: V \to W$ is a <u>morphism</u> if for every $P \in V$, there is a (Zariski) open $U \subseteq V$ containing P and homogeneous

$$F_1, \dots, F_{m+1}$$

s.t. the map φ_U agrees with the map

$$\begin{aligned} U &\to \mathbb{P}^m \\ Q &\mapsto [F_1(Q) : \dots : F_{m+1}(Q)] \end{aligned}$$

Note: For different points, we may need different polynomials, but maybe not (i.e. maybe $U = V$).

Ex: Let $C = V(zx - y^2) \subseteq \mathbb{P}^2$, a plane conic. Consider $C \to \mathbb{P}^1$ defined

$$[x:y:z] \mapsto \begin{cases} [x:y] \text{ if } x \neq 0 & (\text{i.e. on } U_1) \\ [y:z] \text{ if } z \neq 0 & (\text{i.e. on } U_3) \end{cases}$$

This is well-defined: if $x : z \neq 0$, then $y^2 \neq 0$, so

$$[x:y] = [xz : yz] = [y^2 : yz] = [y:z]$$

Def: $\varphi: V \to W$ is an <u>isomorphism</u> if there is an inverse morphism $W \to V$.

Ex: $V(xz - y^2) \to \mathbb{P}^1$ is an isomorphism w/ inverse

$$[s:t] \to [s^2 : st : t^2].$$

($[s^2:st]=[st:t^2]$ when $s,t\neq 0$)

Projective change of coordinates

let $T: \overset{\mathbb{A}^{n+1}}{\overset{\shortparallel}{k^{n+1}}} \longrightarrow \overset{\mathbb{A}^{n+1}}{\overset{\shortparallel}{k^{n+1}}}$ be a linear change of coordinates (as vector spaces). Then $T(0)=0$, and T takes lines through the origin to lines through the origin, so T induces a map $\mathbb{P}^n \longrightarrow \mathbb{P}^n$, called a <u>projective change of coordinates</u>.

i.e. $T=[F_1:F_2:\dots:F_{n+1}]$ where the F_i are linearly independent forms of degree one. We can represent T by an $(n+1)\times(n+1)$ invertible matrix.

(think of the variables as a natural basis.)

Note: T and λT induce the same map on $\mathbb{P}^n$, so the change of coordinates correspond to $\underbrace{GL(n+1,k)}_{\substack{\text{invertible} \\ (n+1)\times(n+1) \text{ matrices}}}/\sim$ where $M\sim\lambda M$, $\lambda\neq 0$.

This is a group under multiplication (i.e. composition) and is called the <u>projective general linear group</u>, denoted $PGL(n+1,k)$.

In fact, these are the only automorphisms of $\mathbb{P}^n$. We won't show this, but you can use complex analysis to try the $k=\mathbb{C}$, $n=2$ case.

Def: $V, W\subseteq\mathbb{P}^n$ algebraic sets are <u>projectively equivalent</u> if $\exists$ a projective change of coordinates that restricts to an isomorphism $V\longrightarrow W$.

Ex: $V(x)$ and $V(y) \subseteq \mathbb{P}^2$ are projectively equivalent:

$[x:y:z] \mapsto [y:x:z]$ is the corr. change of coordinates, w/ matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Similarly, $V(x)$ and $V(y-x)$ are proj. equiv:

$$[x:y:z] \longmapsto [y-x:y:z]$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Some properties of proj. change of coordinates (exer)

$T: \mathbb{P}^n \to \mathbb{P}^n$ a proj. change of coords w/ $T = (T_1, \dots, T_{n-1})$

1.) $V = V(F_1 \dots, F_r)$, F_i homogeneous, then $T^{-1}(V) = V(F_1(T_1, \dots), \dots, F_{n+1}(T_1, \dots))$

2.) T induces isomorphisms $\Gamma_h(V) \to \Gamma_h(T^{-1}(V))$, $k(V) \to k(T^{-1}(V))$, $\mathcal{O}_{T(P)}(V) \to \mathcal{O}_P(T^{-1}(V))$

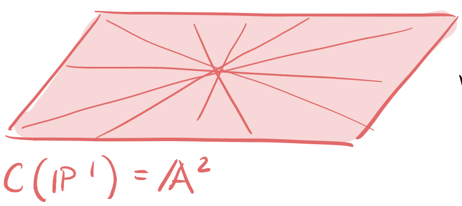
3.) If $W \subseteq \mathbb{P}^n$ is a linear subvariety, $\exists$ T s.t.

$$T^{-1}(W) = V(x_{d+2}, \dots, x_{n+1}), \text{ where } d = \dim W.$$

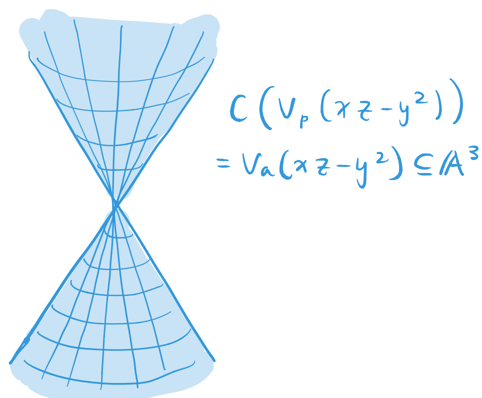
(convention: $d=-1$ if $V = \emptyset$)

Caution: Two affine algebraic sets are isomorphic $\iff$ their coordinate rings are isomorphic. However, " $\Rightarrow$ " doesn't hold in the projective case:

Ex: $V(xz-y^2) \subseteq \mathbb{P}^2$ and $\mathbb{P}^1$ are isomorphic, but their homog. coordinate rings $k[x,y,z]/(xz-y^2)$ and $k[s,t]$ are not (the former is not a UFD). This is because their affine cones aren't isomorphic:



VS.



Projective equivalence does guarantee an isomorphism between homog. coordinate rings.